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A METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS. SECOND PAPER.

By P. A. Lambert.

The object of this paper is to apply to linear partial differential equations the method of solution applied to ordinary linear differential equations in the paper entitled "A Method of Solving Linear Differential Equations" published in the *Annals of Mathematics*, July, 1910.

Let the given differential equation be

$$f\left(x,\, y,\, u,\, \frac{\partial u}{\partial x},\, \frac{\partial u}{\partial y},\, \cdots\, \frac{\partial^n u}{\partial y^n}\right) = 0. \tag{1}$$

The method of solution proposed consists of the following steps:

(*a*) Break up the function f into two parts, one of which, f_1, equated to zero gives a differential equation which may be readily solved, and introduce a parameter t as a factor of the second part f_2, so that the given equation, $f_1 + f_2 = 0$, is replaced by

$$f_1 + tf_2 = 0. \tag{2}$$

(*b*) Assume that the series

$$u = u_0 + u_1 t + u_2 t^2 + u_3 t^3 + \cdots, \tag{3}$$

where $u_0, u_1, u_2, u_3, \cdots$ are undetermined functions of x and y, makes equation (2) an identity. Substitute the expression (3) in equation (2) and determine these functions by solving the differential equations formed by equating to zero the coefficients of successive powers of t in the resulting identity.

(*c*) Substitute these values of $u_0, u_1, u_2, u_3, \cdots$ in (3), and replace t by unity. Then see if

$$u = u_0 + u_1 + u_2 + u_3 + \cdots \tag{4}$$

is convergent and satisfies equation (1).

This method of solving differential equations will be called the parametric method.

The results obtained by the application of this method are not new. The actual solution of an ordinary linear differential equation by the

parametric method is simpler in theory and decidedly less laborious than by the method which assumes the solution to be

$$y = \sum_{r=0}^{r=\infty} A_r x^{m+rs}$$

and requires the determination of the constants A_r, m and s.

The method of solution of linear ordinary differential equations outlined by Schlesinger* is practically identical with a special application of the parametric method. In the application of the method outlined by Schlesinger, which was established by Caqué† by studying equations of finite differences, the given differential equation,

$$f\left(x, y, \frac{dy}{dx}, \cdots \frac{d^n y}{dx^n}\right) = 0$$

is broken up into

$$f_1 - f_2 = p(x) \quad \text{or} \quad D_1(y) - D_2(y) = p(x),$$

where D_1 must contain the derivative of highest order. The solution of $D_1(y) = 0$ is called u_0 and the solution of the given equation is written $y = u_0 + u$, so that

$$D_1(u_0 + u) = D_2(u_0 + u) + p(x),$$

and u must be the principal integral of

$$D_1(u) - D_2(u) = F_0(x),$$

where $F_0(x)$ represents the known function

$$p(x) + D_2(u_0).$$

If u_1 is the principal integral of

$$D_1(u) = F_0(x)$$

and $u = u_1 + v$, v must be the principal integral of

$$D_1(v) = D_2(v) + F_1(x)$$

where

$$F_1(x) = D_2(u_1).$$

Repeated application of this process gives

$$y = u_0 + u_1 + u_2 + u_3 + \cdots,$$

*Handbuch der Theorie der linearen Differentialgleichungen, pp. 370–377.
†Liouville's Journal, Series II, Vol. 9, p. 185.

which is proved to be convergent and a solution of the given differential equation.

In applying the parametric method the given differential equation is replaced by

$$f_1 + tf_2 = 0,$$

where f_1 is selected so that (1) the equation $f_1 = 0$ can be integrated and (2) the resulting series shall be convergent. It is frequently advantageous to select f_1 so that it will not contain the highest derivative of the given equation.

Schwarz* discusses the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + au = 0,$$

and Darboux† discusses the partial differential equation

$$\frac{\partial^2 u}{\partial x\, \partial y} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu,$$

where a, b, c, are functions of x and y, by a method of successive approximations which is essentially the same as the parametric method. However, the use of the parameter, which characterizes my method of solution, does not seem to occur in the literature of differential equations. Moreover, the parametric method seems better adapted to the actual determination of the solution of the partial differential equation than the method of successive approximations as used by Schwarz and Darboux.

The parametric method will be exemplified by applying it to several examples.

In Example I the method is applied to a first order equation to throw the method into prominence. Example II is the second order differential equation of fundamental importance in mathematical physics and the theory of functions. The solution containing two arbitrary functions is found. Examples III, IV and V show how to find particular integrals under different conditions

The differential equations of the first five examples have constant coefficients. In Example VI, the differential equation of Euler and Poisson,‡ of importance in mathematical physics and differential geometry, the coefficients are functions of x and y.

*Abhandlungen, Vol. I, pp. 241–265.

† Théorie générale des surfaces, vol. IV, pp. 353–367.

‡ Darboux, Théorie des surfaces, vol. II, p. 54.

Example I. $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=nu.$$

Replace the equation by

$$\frac{\partial u}{\partial y}-nu+\frac{\partial u}{\partial x}t=0$$

and assume

$$u=u_0+u_1t+u_2t^2+u_3t^3+\cdots.$$

There results

$$\left.\begin{array}{c}\dfrac{\partial u_0}{\partial y}\\[2ex] -nu_0\end{array}\right|+\left.\begin{array}{c}\dfrac{\partial u_1}{\partial y}\\[2ex] -nu_1\\[2ex] +\dfrac{\partial u_0}{\partial x}\end{array}\right|t+\left.\begin{array}{c}\dfrac{\partial u_2}{\partial y}\\[2ex] -nu_2\\[2ex] +\dfrac{\partial u_1}{\partial x}\end{array}\right|t^2+\cdots\equiv 0.$$

Solving the differential equations obtained by equating to zero the coefficients of powers of t in this identity,

$$u_0=e^{ny}\varphi(x),$$

$$u_1=-e^{ny}\varphi'(x)y,$$

$$u_2=e^{ny}\varphi''(x)\frac{y^2}{2!},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Substituting in the value of u and making t unity,

$$u=e^{ny}\left[\varphi(x)-\varphi'(x)y+\varphi''(x)\frac{y^2}{2!}-\cdots\right],$$

whence by Taylor's series

$$u=e^{ny}\varphi(x-y),$$

the integral of the given differential equation containing an arbitrary function.

Example II. $$\frac{\partial^2 u}{\partial x^2}+a^2\frac{\partial^2 u}{\partial y^2}=0.$$

Replacing this equation by

$$\frac{\partial^2 u}{\partial x^2}+a^2t\frac{\partial^2 u}{\partial y^2}=0,$$

and assuming

$$u=u_0+u_1t+u_2t^2+u_3t^3+\cdots,$$

there results

$$\frac{\partial^2 u_0}{\partial x^2} + \left.\begin{matrix}\dfrac{\partial^2 u_1}{\partial x^2} \\ + a^2 \dfrac{\partial^2 u_0}{\partial y_2}\end{matrix}\right| t + \left.\begin{matrix}\dfrac{\partial^2 u_2}{\partial x^2} \\ + a^2 \dfrac{\partial^2 u_1}{\partial y^2}\end{matrix}\right| t^2 + \cdots \equiv 0.$$

From the series of differential equations obtained by equating to zero the coefficients of powers of t in this identity may be found

$$u_0 = (A + Bx)\varphi(y),$$

$$u_1 = -a^2\left(A\frac{x^2}{2!} + B\frac{x^3}{3!}\right)\varphi''(y),$$

$$u_2 = a^4\left(A\frac{x^4}{4!} + B\frac{x^5}{5!}\right)\varphi'''(y),$$

$$\cdots\cdots\cdots\cdots$$

Substituting in u and making t unity,

$$u = A\varphi(y) - A\frac{a^2x^2}{2!}\varphi''(y) + A\frac{a^4x^4}{4!}\varphi^{\mathrm{IV}}(y) + \cdots$$
$$+ B\varphi(y)x - B\frac{a^2x^3}{3!}\varphi''(y) + B\frac{a^4x^5}{5!}\varphi^{\mathrm{IV}}(y) - \cdots.$$

Now if $\varphi(y)$ is made e^y in order that $\varphi(y)$ and all its derivatives shall have the same value, and if the arbitrary constant B is replaced by Ca, it is readily seen that

$$u = Ae^y \cos(ax) + Ce^y \sin(ax).$$

It is evident that this solution of the differential equation may be written in the form

$$u = Ae^{ny}\cos\frac{ax}{n} + Ce^{ny}\sin\frac{ax}{n}.$$

If it is desired that u shall contain all the successive derivatives of $\varphi(y)$ the values of u_0, u_1, u_2, u_3, $\cdots$ may be written

$$u_0 = \varphi(y) + Bx\varphi'(y)$$

$$u_1 = -\varphi''(y)\frac{a^2x^2}{2!} - B\varphi'''(y)\frac{a^2x^3}{3!}$$

$$u_2 = \varphi^{\mathrm{IV}}(y)\frac{a^4x^4}{4!} + B\varphi^{\mathrm{V}}(y)\frac{a^4x^5}{5!}$$

$$\cdots\cdots\cdots\cdots$$

Whence

$$u = \varphi(y) + \varphi'(y)Bx - \varphi''(y)\frac{a^2x^2}{2!}$$

$$- \varphi'''(y)B\frac{a^2x^3}{3!} + \varphi^{\mathrm{IV}}(y)\frac{a^4x^4}{4!} + \cdots.$$

The terms of n which have been written are the first five terms of the expansion of a function by Taylor's series provided $B = ai$ or $B = -ai$, where $i = \sqrt{-1}$.

The terms of u which have been written now suggest that

$$u = \varphi_1(y + iax) + \varphi_2(y - iax),$$

which on trial is found to be correct.

Of this solution containing two arbitrary functions the former solution containing two arbitrary constants is a special case.

Example III. $$\frac{\partial^3 u}{\partial x^2 \partial y} - 2\frac{\partial^3 y}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial y^3} = \frac{1}{x^2}.$$

Replacing this equation by

$$\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{1}{x^2} - \left(2\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3}\right)t = 0,$$

and applying the method

$$u_0 = \varphi_1(x) + (A + Bx)\varphi_2(y) + (A_1 + B_1x)\varphi_2'(y) - y \log x.$$

By actual trial it is found that the following four values of u_0, parts of this general expression for u_0,

$$u_0 = \varphi_1(x),$$

$$u_0 = \varphi_2(y) + B_1x\varphi_2'(y),$$

$$u_0 = Bx\varphi_3(y),$$

$$u_0 = -y \log x,$$

lead to solutions of the differential equation.

The solution of the equation, the sum of the solutions corresponding to these four values of u_0, is

$$u = \varphi_1(x) + \varphi_2(x + y) + x\varphi_3(x + y) - y \log x.$$

Example IV. $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = e^{x+2y} + xy.$$

The complementary integral is the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0.$$

Replacing this equation by

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial x} - \left(\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial y}\right)t = 0,$$

we find $u_0 = e^{3x}\varphi(y)$, which suggests that e^{3x} is a factor of the complementary integral.

Transforming the equation by the relation $u = e^{3x}\cdot v$, a partial differential equation is formed of which

$$v = \varphi(x - y)$$

is a solution. It follows that

$$u = e^{3x}\varphi(x - y)$$

is a complementary integral.

If the differential equation which determines the complementary integral is replaced by

$$\left(\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial x}\right)t - \left(\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial y}\right) = 0$$

it is found in like manner that

$$u = e^{3y}\varphi(x - y)$$

is a complementary integral.

These two complementary integrals are not independent. Replacing the differential equation by

$$\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)t - 3\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 0,$$

a third and independent complementary integral is found

$$u = \varphi(x + y).$$

The particular integral corresponding to the term xy is found by applying the method to

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial x} - xy - \left(\frac{\partial^2 u}{\partial y^2} - 3\frac{\partial u}{\partial y}\right)t = 0.$$

To find the particular integral corresponding to the term e^{x+2y} transform the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 3\frac{du}{\partial x} + 3\frac{\partial u}{\partial y} = e^{x+2y}$$

by the relation

$$u = v\cdot e^{x+2y}.$$

The equation in v is

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = 1.$$

A special solution of this equation is $v = -y$, and the corresponding particular integral of the given equation is

$$u = -y\cdot e^{x+2y}.$$

The solution of the given differential equation is the sum of the two independent complementary integrals and the two parts of the particular integral.

Example V. To find a particular integral of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 6\frac{\partial^2 u}{\partial y^2} = x^2 \sin(x+y).$$

Replace the right-hand member by $x^2 e^{ix+iy}$ and transform the resulting equation by the relation

$$u = e^{ix+iy}\cdot v.$$

The particular integral of the equation in v can be found by the method of example IV.

The particular integral of the given equation is the coefficient of i in the corresponding value of u.

So far the parametric method has been applied to linear partial differential equations with constant coefficients. The following example shows that the method may be applied with advantage to equations with variable coefficients.

Example VI. $$\frac{\partial^2 u}{\partial x \partial y} - \frac{a}{x-y}\frac{\partial u}{\partial x} + \frac{b}{x-y}\frac{\partial u}{\partial y} = 0.$$

Replacing this equation by

$$a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} = (x - y)\frac{\partial^2 u}{\partial x \partial y}t$$

and assuming that

$$u = u_0 + u_1 t + u_2 t^2 + u_3 t^3 + \cdots,$$

it follows at once that

$$u_0 = \varphi(ay + bx).$$

Solutions of the given equations which are homogeneous polynomials are found by giving u_0 the successive values

$$u_0 = ay + bx,$$

$$u_0 = (ay + bx)^2,$$

$$u_0 = (ay + bx)^3,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The given equation may be written

$$\left(x\frac{\partial^2 u}{\partial x \partial y} + b\frac{\partial u}{\partial y}\right) - \left(y\frac{\partial^2 u}{\partial x \partial y} + a\frac{du}{\partial x}\right) = 0.$$

A solution of the first part of this equation placed equal to zero is

$$u = x^{-b}\varphi_1(y);$$

a solution of the second part placed equal to zero is

$$u = y^{-a}\varphi_2(x).$$

Hence a solution of the given equation is

$$u = x^{-b}y^{-a},$$

from which we may get the more general solution

$$u = (x + m)^{-b}(y + m)^{-a},$$

where m is an arbitrary constant.

The given differential equation may be replaced by either of the equations

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{a}{x - y}\frac{\partial u}{\partial x} + \frac{b}{x - y}\frac{\partial u}{\partial y}t = 0,$$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{b}{x - y}\frac{\partial u}{\partial y} - \frac{a}{x - y}\frac{\partial u}{\partial x}t = 0.$$

From the first equation a special value of u_0 is

$$u_0 = \frac{-(x-y)^{1-a}}{1-a}.$$

The use of this value of u_0 suggests that $(x-y)^{1-a}$ is a factor of the solution of the given equation. In like manner from the second equation

$$u_0 = \frac{-(x-y)^{1-b}}{1-b},$$

which suggests that $(x-y)^{1-b}$ is a factor of the solution of the given equation.

From these suggestions it is inferred that $(x-y)^{1-a-b}$ is a factor of the solution of the given equation. This inference is correct, for the relation

$$u = (x-y)^{1-a-b} \cdot v$$

transforms the given equation into

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{1-b}{x-y}\frac{\partial v}{\partial x} + \frac{1-a}{x-y}\frac{\partial v}{\partial y} = 0.$$

By the preceding paragraph a solution of this equation is

$$v = (x+m)^{a-1}(y+n)^{b-1}.$$

The corresponding solution of the given equation is

$$u = (x+m)^{a-1}(y+m)^{b-1}(x-y)^{1-a-b}.$$

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